

Twisted Superalgebras and Cohomologies of the $N = 2$ Superconformal Quantum Mechanics

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Abstract

We prove that the invariance of the $N = 2$ superconformal quantum mechanics is controlled by subalgebras of a given twisted superalgebra

made of 6 fermionic (nilpotent) generators and 6 bosonic generators (including a central charge). The superconformal quantum mechanics actions are invariant under this quite large twisted superalgebra. On the other hand, they are fully determined by a subalgebra with only 2 fermionic and 2 bosonic (the central charge and the ghost number) generators. The invariant actions are Q_i -exact ($i = 1, 2, \dots, 6$), with a $Q_{i'}$ -exact ($i' \neq i$) antecedent for all 6 fermionic generators. It follows that the superconformal quantum mechanics actions with Calogero potentials are uniquely determined even if, in its bosonic sector, the twisted superalgebra does not contain the one-dimensional conformal algebra $sl(2)$, but only its Borel subalgebra. The general coordinate covariance of the non-linear sigma-model for the $N = 2$ supersymmetric quantum mechanics in a curved target space is fully implied only by its worldline invariance under a pair of the 6 twisted supersymmetries. The transformation connecting the ordinary and twisted formulations of the $N = 2$ superconformal quantum mechanics is explicitly presented.

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1 Introduction.

Both the ordinary and the conformal $N = 2$ supersymmetric quantum mechanical models describe interesting dynamics. The $N = 2$ supersymmetry has applications to solvable potentials and the motion of particles in certain gravitational backgrounds [1]–[9]. It is an appropriate framework to describe topological invariants of various target-spaces for the propagating bosons [10]. The worldline $N = 2$ global supersymmetry has a natural extension as a superconformal algebra ($N = 2$ SCA) [11].

The interpretation of the $N = 2$ supersymmetric quantum mechanics as a topological quantum field theory derives from the invariance under a twisted superalgebra, whose nilpotent fermionic generators define a BRST-type cohomology [12, 13, 14, 15].

We construct here a twisted superalgebra, called “twisted $N = 2$ SCA”, acting on a set of $(1, 2, 1)$ supermultiplets $(X_\mu(t); \Psi_\mu(t), \bar{\Psi}_\mu(t); b_\mu(t))$. For any given μ , $1 \leq \mu \leq d$, $X_\mu(t)$ is a propagating boson, $\Psi_\mu(t)$, $\bar{\Psi}_\mu(t)$ are its two anticommuting supersymmetric partners and $b_\mu(t)$ is a commuting auxiliary field. Such multiplets are called balanced because they contain an equal number of bosonic and fermionic component fields. For us the expression “twisted superalgebra” just means that its fermionic generators are all nilpotent; the relation between the twisted $N = 2$ SCA and the ordinary $N = 2$ SCA is that the invariance of an action under the twisted $N = 2$ SCA implies the invariance under the ordinary $N = 2$ SCA and vice-versa. In fact, the construction extends that of [15] from global supersymmetries to the case of supersymmetries whose generators carry an explicit dependence on the t coordinate.

The ordinary one-dimensional $N = 2$ superconformal algebra of the $(1, 2, 1)$ supermultiplet is the simple Lie superalgebra $sl(2|1)$ [16], with 4 bosonic and 4 fermionic generators. It contains as a subalgebra, in its bosonic sector, the $sl(2)$ algebra which defines the one-dimensional Conformal Quantum Mechanics [17].

The existence of the twisted $N = 2$ SCA is interesting for at least two reasons. It selects invariants as local cocycles and, furthermore, it contains quite a small subalgebra which is sufficient to determine the full superconformal invariance.

To build the twisted $N = 2$ SCA, we can construct at first a global twisted $N = 2$ superalgebra with 4 fermionic generators, acting on the $(1, 2, 1)$ supermultiplet, the two worldline “twisted scalar supersymmetries” Q and \bar{Q} and the two “twisted vector supersymmetries” Q_V and \bar{Q}_V . Then, one can complement these 4 fermionic nilpotent generators by two extra nilpotent ones which carry the explicit t -dependence, the “twisted conformal supersymmetries” Q_C and \bar{Q}_C .

The \mathbf{Z}_2 -graded anticommutators of 6 fermionic generators close on a set of 6 bosonic generators, one of them being a central charge. Due to the presence of the central charge, the twisted $N = 2$ SCA is not a simple Lie superalgebra as the ordinary $N = 2$ SCA.

The generators of the twisted $N = 2$ SCA can be realized as 4×4 supermatrices with 2×2

even/odd blocks, whose entries are t -dependent differential operators. This provides a so-called D -module representation.

The twisted $N = 2$ SCA induces the same invariant actions as the ordinary $N = 2$ SCA. However, it does not contain the conventional conformal $sl(2)$ symmetry as a bosonic subalgebra. Rather, only a Borel subalgebra of the conformal $sl(2)$ belongs to the twisted $N = 2$ SCA.

We found that imposing the invariance under a small subalgebra is sufficient to guarantee the invariance under the full twisted $N = 2$ SCA and, consequently, the ordinary $N = 2$ SCA.

The minimal superalgebra which enforces the $N = 2$ superconformal invariance is made of only 2 fermionic generators (either Q and \bar{Q}_C or \bar{Q} and Q_C) and 2 bosonic generators (the central charge c with a fixed coefficient and the ghost number N_{gh}). Therefore, the whole superconformal invariance is obtained “for free”, with the extra generators regarded as “accidental” symmetries of this minimal set-up.

This intriguing result offers a new perspective for investigating the conformal properties of the supersymmetric models, since it suggests that one can replace the demand of the conformal symmetry by that of a much smaller symmetry ⁶.

Conformally invariant topological theories such as the topological quantum field theory toy model [19] and the Calogero models are thus very economically defined by simply demanding the invariance under the 2 supercharges Q and \bar{Q}_C , instead of imposing the complete worldline superconformal symmetry. New Lagrangians with higher-order interactions among Fermi fields, involving fields that are one-dimensional analogues of the Ramond fields, can also be constructed.

Further results in this paper can be summarized as follows. The invariant actions I are Q_i -exact ($i = 1, 2, \dots, 6$), with a local $Q_{i'}$ -exact ($i' \neq i$) antecedent for all 6 fermionic generators of the twisted $N = 2$ SCA ($I = \int dt Q_i Q_{i'} Z_{ii'}$).

If one relaxes the condition of conformal invariance, the general coordinate covariance of the non-linear sigma-model for the $N = 2$ supersymmetric quantum mechanics in a curved target space is fully implied by its worldline invariance under the action of only two of the above mentioned 6 supercharges, either Q and \bar{Q}_V or \bar{Q} and Q_V . (The invariance under the 2 extra supercharges is automatically obtained). One can search for target metrics such that the action is superconformally invariant [8].

An invertible complex transformation relates the component fields and the free actions of the ordinary and the twisted $N = 2$ superconformal algebra.

The paper is organized as follows. In Section 2 we introduce the twisted $N = 2$ superconformal algebra and discuss the relevant subalgebras. In section 3 we investigate the different cohomologies which induce superconformally invariant actions (both free and in the presence

⁶ These considerations have analogies in the $N = 4$, $d = 4$ super-Yang-Mills theory, where the superconformal Yang-Mills supersymmetry with its 32 supersymmetric generators is implied by a much smaller superalgebra, with only 4 scalar twisted nilpotent supercharges [18].

of an interacting potential) in the case of a flat target-space. In Section 4 we discuss the implications of the cohomologies for the covariance of a curved target manifold. In Section 5 the construction of higher-order Fermi interactions is pointed out. The explicit transformation relating the ordinary and the twisted formulations of the $N = 2$ SCA is presented in Section 6. In the Conclusion we make a comparison between our observations concerning these simple supersymmetric quantum mechanical systems and the intriguing results that have been recently observed in higher dimensional supersymmetric quantum field theories. We also discuss the future perspectives of our work.

2 The twisted $N = 2$ superconformal algebra.

We recall at first some known facts. Let \mathcal{M}_d be a d -dimensional manifold, locally parametrized by the $1 \leq \mu \leq d$ coordinates X^μ , with metric $g_{\mu\nu}(X)$, Christoffel symbols $\Gamma_{\mu\nu,\rho}$ and Riemann tensor $R_{\mu\nu,\rho\sigma}$. The target-space reparametrization covariant action with worldline $N = 2$ supersymmetry is expressed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \bar{\Psi}^\mu(g_{\mu\nu}\dot{\Psi}^\nu + \Gamma_{\mu,\rho\sigma}\dot{X}^\rho\Psi^\sigma) + \frac{1}{4}R_{\mu\nu\rho\sigma}\bar{\Psi}^\rho\Psi^\sigma\bar{\Psi}^\nu\Psi^\mu. \quad (1)$$

t parametrizes the worldline and $\dot{\Phi} \equiv \frac{d\Phi}{dt}$. The possibility of choosing any given parametrization is obvious since we are working with a one-dimensional parametrization.

The t -dependent coordinates $X^\mu(t)$ are bosons, while $\Psi^\mu(t)$ and $\bar{\Psi}^\mu(t)$ are fermions. Using an auxiliary field $b^\mu(t)$, one can express \mathcal{L} as [14]

$$\mathcal{L} = g_{\mu\nu}b^\nu b^\mu + b^\mu(-g_{\mu\nu}\dot{X}^\nu + \Gamma_{[\mu,\rho]\sigma}\bar{\Psi}^\rho\Psi^\sigma) + \partial_\rho g_{\mu\nu}\bar{\Psi}^\mu\Psi^\rho X^\nu + \bar{\Psi}^\mu(g_{\mu\nu}\dot{\Psi}^\nu + \Gamma_{\mu,\rho\sigma}\dot{X}^\rho\Psi^\sigma). \quad (2)$$

The general covariance in the curved target-space with coordinates X^μ is explicit for the action (1). However, such an important invariance is only enforced after the elimination from the action (2) of the auxiliary fields b^μ via their algebraic equations of motion $b_\mu = g_{\mu\nu}\dot{X}^\nu - \Gamma_{[\mu,\rho]\sigma}\bar{\Psi}^\rho\Psi^\sigma$. These equations show that the “on-shell” b^μ -replaced fields are not vectors. The action (2) is not invariant under target-space general coordinates transformations due to the fact that it inherently involves the Christoffel symbols $\Gamma_{[\mu,\rho]\sigma}$ and that it is not possible to redefine the b^μ fields in order to absorb this dependence. On the other hand, when the auxiliary fields are present, all (twisted or untwisted) supersymmetry transformations close “off-shell”. After the elimination of the b^μ fields via their equations of motion, the supersymmetries only close modulo some fermionic equations of motion. All this boils down to the fact that $X, \Psi, \bar{\Psi}, b$ is a balanced multiplet (for simplicity, from now on we drop, when not necessary, the μ suffix), while $X, \Psi, \bar{\Psi}$ is an unbalanced multiplet, that is it contains an unequal number of bosonic and fermionic component fields. These intriguing facts about what is happening in the presence or after eliminating the auxiliary fields are however not troublesome when the fermionic twisted

generators are realized as nilpotent generators [14]. From the point of view of studying the world-line supersymmetry, the action (2) is more suitable.

The balanced quantum mechanical supersymmetric multiplet is thus made of d independent $(1, 2, 1)$ supermultiplets, whose fields are target-space vectors

$$X^\mu(t), \Psi^\mu(t), \bar{\Psi}^\mu(t), b^\mu(t), \quad 1 \leq \mu \leq d. \quad (3)$$

In the flat case the metric is $g_{\mu\nu}(X) = \eta_{\mu\nu}$ and the Lagrangian is simply given by

$$\mathcal{L} = b^\mu \eta_{\mu\nu} \dot{b}^\nu - b^\mu \eta_{\mu\nu} \dot{X}^\nu + \bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu \sim -\frac{1}{4} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + \bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu. \quad (4)$$

Two important bosonic charges are conserved and compatible with all fermionic transformations, the field dimension (also known as “engineering dimension”) and the ghost number. Their values for the components of the balanced multiplet are, respectively, $(-\frac{1}{2}, 0, 0, \frac{1}{2})$ and $(0, 1, -1, 0)$. The action has ghost number zero and is dimensionless if we assume the world-line parameter t to possess the engineering dimension -1 .

In the following we will show the existence of other fermionic invariances of the free Lagrangian. We will check which interactions can preserve at least some of them.

2.1 Construction and presentation of the twisted $N = 2$ SCA.

The 6 nilpotent fermionic generators can be divided into one pair of worldline scalar twisted supersymmetry operators Q, \bar{Q} , one pair of worldline vector twisted supersymmetry operators Q_V, \bar{Q}_V and one pair of worldline scalar special twisted supersymmetry operators Q_C, \bar{Q}_C . $Q, \bar{Q}, Q_V, \bar{Q}_V$ are constructed with the prescription of [15], while Q_C, \bar{Q}_C are determined by demanding explicit t -dependence and compatibility with ghost number and engineering dimension.

These 6 operators act on the component fields according to the transformations

$$\begin{aligned}
QX &= \Psi, & \bar{Q}X &= \bar{\Psi}, \\
Q\Psi &= 0, & \bar{Q}\Psi &= -b, \\
Q\bar{\Psi} &= b, & \bar{Q}\bar{\Psi} &= 0, \\
Qb &= 0, & \bar{Q}b &= 0. \\
\\
Q_V X &= \bar{\Psi}, & \bar{Q}_V X &= \Psi, \\
Q_V \Psi &= -b + \dot{X}, & \bar{Q}_V \Psi &= 0, \\
Q_V \bar{\Psi} &= 0, & \bar{Q}_V \bar{\Psi} &= b - \dot{X}, \\
Q_V b &= \dot{\Psi}, & \bar{Q}_V b &= \dot{\bar{\Psi}}. \\
\\
Q_C X &= t\Psi, & \bar{Q}_C X &= t\bar{\Psi}, \\
Q_C \Psi &= 0, & \bar{Q}_C \Psi &= -tb + \bar{\lambda}X, \\
Q_C \bar{\Psi} &= tb - \lambda X, & \bar{Q}_C \bar{\Psi} &= 0, \\
Q_C b &= \lambda\Psi, & \bar{Q}_C b &= \bar{\lambda}\bar{\Psi}.
\end{aligned} \tag{5}$$

Until now the real parameters $\lambda, \bar{\lambda}$ are arbitrary. We are however forced to set $\bar{\lambda} = \lambda$ in order to eliminate unwanted anticommutation relations (the presence of a t -multiplication operator) arising from $\{Q_C, \bar{Q}_C\}$. This setting guarantees that $\{Q_C, \bar{Q}_C\} = 0$.

Therefore, the only non-vanishing anticommutators are

$$\begin{aligned}
\{Q, Q_V\} &= H, & \{\bar{Q}, \bar{Q}_V\} &= H, \\
\{Q, \bar{Q}_C\} &= c, & \{\bar{Q}, Q_C\} &= -c, \\
\{Q_C, Q_V\} &= S, & \{\bar{Q}_C, \bar{Q}_V\} &= \bar{S}, \\
\{Q_V, \bar{Q}_C\} &= Z, & \{\bar{Q}_V, Q_C\} &= \bar{Z}.
\end{aligned} \tag{6}$$

A central charge c ,

$$c = \lambda \mathbf{1}, \tag{7}$$

has arisen from the anticommutators of Q with \bar{Q}_C and \bar{Q} with Q_C .

The action on the component fields of the bosonic operators $H, S, \bar{S}, Z, \bar{Z}$ is as follows

$$\begin{aligned}
H &= \frac{d}{dt}, \\
S &= t \frac{d}{dt} + \Delta, \\
\bar{S} &= -t \frac{d}{dt} + \bar{\Delta},
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
ZX &= Z\bar{\Psi} = Zb = 0, & Z\Psi &= \bar{\Psi}, \\
\bar{Z}X &= \bar{Z}\Psi = \bar{Z}b = 0, & \bar{Z}\bar{\Psi} &= -\Psi,
\end{aligned} \tag{9}$$

where $\Delta, \bar{\Delta}$ in (8) act as diagonal operators:

$$\begin{aligned}\Delta X &= -\lambda X, & \bar{\Delta} X &= \lambda X, \\ \Delta b &= (1 - \lambda)b, & \bar{\Delta} b &= (\lambda - 1)b, \\ \Delta \Psi &= (1 - \lambda)\Psi, & \bar{\Delta} \Psi &= \lambda\Psi, \\ \Delta \bar{\Psi} &= -\lambda\bar{\Psi}, & \bar{\Delta} \bar{\Psi} &= (\lambda - 1)\bar{\Psi}.\end{aligned}\tag{10}$$

The 12 operators entering (6) are closed under (anti)commutation relations, so that

$$\mathcal{G}^\sharp \equiv \{Q, \bar{Q}, Q_V, \bar{Q}_V, Q_C, \bar{Q}_C, H, c, S, \bar{S}, Z, \bar{Z}\}\tag{11}$$

is a Lie superalgebra.

The non-vanishing commutators involving the even operators of \mathcal{G}^\sharp are

$$\begin{aligned}[H, S] &= [\bar{S}, H] = H, \\ [S, Z] &= [\bar{S}, Z] = Z, \\ [\bar{Z}, S] &= [\bar{Z}, \bar{S}] = \bar{Z}, \\ [\bar{Z}, Z] &= S + \bar{S}.\end{aligned}\tag{12}$$

The non-vanishing commutators between even and odd generators of \mathcal{G}^\sharp are

$$\begin{aligned}[H, Q_C] &= Q, & [H, \bar{Q}_C] &= \bar{Q}, \\ [S, Q] &= -Q, & [S, \bar{Q}_V] &= -\bar{Q}_V, & [S, \bar{Q}_C] &= \bar{Q}_C, \\ [\bar{S}, \bar{Q}] &= \bar{Q}, & [\bar{S}, Q_V] &= Q_V, & [\bar{S}, Q_C] &= -Q_C, \\ [Z, Q] &= -\bar{Q}, & [Z, \bar{Q}_V] &= -Q_V, & [Z, Q_C] &= -\bar{Q}_C, \\ [\bar{Z}, \bar{Q}] &= Q, & [\bar{Z}, Q_V] &= \bar{Q}_V, & [\bar{Z}, \bar{Q}_C] &= Q_C.\end{aligned}\tag{13}$$

From the action of the \mathcal{G}^\sharp operators on the component fields one can immediately write down a D -module representation of \mathcal{G}^\sharp in terms of 4×4 supermatrices, as displayed in the Appendix A.

The Lie superalgebra \mathcal{G}^\sharp is compatible with the following assignment for the scaling dimensions of the component fields and of the generators (we set for the worldline coordinate t the dimension $[t] = -1$). For the component fields we have

$$[\Psi] = x + z, \quad [\bar{\Psi}] = x + 1 - z, \quad [b] = x + 1,\tag{14}$$

where $x = [X]$ is an arbitrary parameter.

For the fermionic generators we have

$$[\bar{Q}] = [Q_V] = -[Q_C] = 1 - z, \quad [\bar{Q}_V] = -[\bar{Q}_C] = z,\tag{15}$$

with $z = [Q]$ an arbitrary parameter.

So far the parameters λ, x, z are arbitrary. On the other hand λ and x have to be fixed by the requirement of scale and conformal invariance. Indeed, x has to be set

$$x = -\frac{1}{2} \quad (16)$$

in order to make dimensionless the free kinetic action. The parameter λ has to be fixed

$$\lambda = \frac{1}{2} \quad (17)$$

in order to guarantee the invariance under Q_C of the free kinetic action.

Without loss of generality, the parameter z can be fixed to be $z = \frac{1}{2}$ to allow $\Psi, \bar{\Psi}$ having the same dimension.

The combinations $S \pm \bar{S}$ are particularly important. $S + \bar{S}$ is the ghost number operator

$$N_{gh} := S + \bar{S}, \quad (18)$$

while

$$D := \frac{1}{2}(S - \bar{S}) = t \frac{d}{dt} + d_s \quad (19)$$

contains the diagonal matrix d_s with the engineering or scaling dimension of the component fields. The ghost number and the scale dimensions are given by

	N_{gh}	d_s
X	0	$-\frac{1}{2}$
b	0	$\frac{1}{2}$
Ψ	1	0
$\bar{\Psi}$	-1	0

(20)

The $sl(2)$ conformal algebra of the one-dimensional conformal quantum mechanics acts with the following D -module unidimensional transformations on an arbitrary s -dimensional field $\Phi_s(t)$ (in our case $s = -\frac{1}{2}$ for X , $s = 0$ for Ψ and $\bar{\Psi}$, $s = \frac{1}{2}$ for b):

$$\begin{aligned} L_{-1} &= \frac{d}{dt}, \\ L_0 &= t \frac{d}{dt} + s, \\ L_1 &= -t^2 \frac{d}{dt} - 2st. \end{aligned} \quad (21)$$

The non-vanishing commutators are

$$\begin{aligned} [L_0, L_{\pm 1}] &= \pm L_{\pm 1}, \\ [L_1, L_{-1}] &= 2L_0. \end{aligned} \quad (22)$$

The conformal $sl(2)$ is not a bosonic subalgebra of \mathcal{G}^\sharp . \mathcal{G}^\sharp possesses an $sl(2)$ subalgebra given by N_{gh}, Z, \bar{Z} . This $sl(2)$ subalgebra does not generate the conformal transformations on the component fields. On the other hand, \mathcal{G}^\sharp possesses the Borel subalgebra of $sl(2)$. Indeed, the subalgebra $\{D, H\}$, with D introduced in (19), is identified with $\{L_0, L_{-1}\}$, so that we can identify $D \equiv L_0$ and $H \equiv L_{-1}$. One should note that D acts on the component fields with the correct assignment of their scale dimensions.

It is quite remarkable, as we will discuss later, that the invariance under this subalgebra is sufficient to determine the conformally invariant actions in quantum mechanics, both for the bosonic and the fermionic sectors.

Even more remarkable, the invariance under just 2 twisted fermionic generators, together with the requirement of the vanishing of the ghost number, is sufficient to determine the superconformally invariant actions.

We denote as “ \mathcal{G}_{min}^\sharp ” the minimal \mathcal{G}^\sharp subalgebra which, imposed as invariance of the action, determines the full set of $N = 2$ superconformal invariances. \mathcal{G}_{min}^\sharp is given by

$$\mathcal{G}_{min}^\sharp = \{Q, Q_C, c, N_{gh}\}. \quad (23)$$

Imposing the \mathcal{G}_{min}^\sharp invariance is a very economical way to impose the full $N = 2$ superconformal invariance.

It is convenient to present here a list of subalgebras for \mathcal{G}^\sharp , which can be relevant for different purposes. We have, for instance,

$$\begin{aligned} & \{Q, \bar{Q}_C, c\}, \\ & \{Q, \bar{Q}_C, c, S\}, \\ & \{Q, \bar{Q}, \bar{Q}_C, c, S, H\}, \\ & \{Q, \bar{Q}, Q_V, \bar{Q}_C, c, S, H, Z\}, \\ & \{Q, \bar{Q}, Q_C, \bar{Q}_C, c, S, H, Z\}, \\ & \{Q, \bar{Q}, Q_C, \bar{Q}_V, Q_V, c, S, H, \bar{S}\}. \end{aligned} \quad (24)$$

As we will see in the next Section, Q - and Q_V -invariant actions are not necessarily Q_C -invariant, while Q - and Q_C -invariant actions are necessarily Q_V -invariant.

An important subalgebra of \mathcal{G}^\sharp is denoted by \mathcal{B} . Its 5 generators are

$$\mathcal{B} \equiv \frac{1}{2}(S - \bar{S}), H, Z + \bar{Z}, Q + Q_V - \bar{Q}, \bar{Q} - \bar{Q}_V - Q. \quad (25)$$

As discussed in Section 6, \mathcal{B} coincides with the Borel subalgebra of the $sl(2|1)$ superalgebra.

After having defined the twisted superalgebra \mathcal{G}^\sharp with its set of nilpotent fermionic generators, we are now looking for actions which are invariant under the full \mathcal{G}^\sharp or some of its subalgebras. We will restrict ourselves to the case of a standard kinetic term for the bosons X^μ , namely with a Lagrangian of the form $\mathcal{L} \sim \int dt g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + \dots$

3 Invariant actions in the flat target-space.

3.1 The free supersymmetric Lagrangian.

3.1.1 The invariance under Q and Q_V .

Let us enforce the Q and Q_V invariance of the action $\mathcal{S} = \int dt \mathcal{L}$ and let us assume the free action to be non-dimensional. It will therefore be uniquely defined, with the dimensionality of X^μ fixed to be $x = -\frac{1}{2}$.

We have

$$\mathcal{S} = QQ_V \int dt (\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu) = \int dt Q \left(\bar{\Psi}^\mu \eta_{\mu\nu} (b^\nu - \dot{X}^\nu) \right) = \int dt \left(b^\mu \eta_{\mu\nu} b^\nu - b^\mu \eta_{\mu\nu} \dot{X}^\nu + \bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu \right). \quad (26)$$

By eliminating the b^μ fields through their algebraic equations of motion one gets the supersymmetric Lagrangian $\mathcal{L} \sim -\frac{1}{4} \dot{X}^\mu \dot{X}^\nu + \bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu$.

The QQ_V -exact term $QQ_V(b^\mu \eta_{\mu\nu} X^\nu)$ has the appropriate dimension. It is however, modulo a pure time-derivative, equal to $QQ_V(\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu)$ and, therefore, it is not independent. Indeed, both terms $b^\mu \eta_{\mu\nu} b^\nu$ and $-b^\mu \eta_{\mu\nu} \dot{X}^\nu + \bar{\Psi}^\mu \dot{\Psi}^\nu$ are separately Q -exact and thus Q -invariant; the Q_V invariance on the other hand fixes the relative coefficients of these terms. The action $\int dt (b^\mu \eta_{\mu\nu} b^\nu - b^\mu \eta_{\mu\nu} \dot{X}^\nu + \bar{\Psi}^\mu \dot{\Psi}^\nu)$ is thus completely determined by requiring the invariance under both Q and Q_V .

This action can also be written as $Q\bar{Q}_V \int dt (\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu)$. It is therefore Q , \bar{Q} , Q_V and \bar{Q}_V invariant. It cannot however be written neither as a $Q\bar{Q}$ -exact term, nor as a $Q_V\bar{Q}_V$ -exact term. This property remains true even in the case of a curved target-space; surprisingly, the $Q\bar{Q}_V$ -invariance automatically implies the target-space covariance.

3.1.2 The Q_C invariance.

We leave for the time being arbitrary the parameter λ entering (5) and we check under which condition the QQ_V -invariant action $QQ_V \int dt (\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu)$ is also Q_C -invariant. We obtain

$$Q_C \mathcal{L} = Q_C QQ_V (\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu) = \frac{d}{dt} (b^\mu \eta_{\mu\nu} X^\nu) + (2\lambda - 1) b^\mu \eta_{\mu\nu} \Psi^\nu. \quad (27)$$

Therefore the Q_C -invariance is ensured provided that $\lambda = \frac{1}{2}$ (see the formula (17)).

Modulo a time derivative one gets, for the Lagrangian,

$$\mathcal{L} = Q_C \bar{Q} \left(\bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu \right). \quad (28)$$

Furthermore one obtains

$$\bar{Q} \left(\bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu \right) = \bar{\Psi}^\mu \eta_{\mu\nu} \dot{b}^\nu \sim -\dot{\bar{\Psi}}^\mu \eta_{\mu\nu} b^\nu = -Q_V b^\mu \eta_{\mu\nu} b^\nu = -\frac{1}{2} Q_V (b^\mu \eta_{\mu\nu} b^\nu). \quad (29)$$

Therefore \mathcal{L} is also, modulo a time derivative, $Q_C Q_V$ -exact,

$$\mathcal{L} = Q_C Q_V \left(\frac{1}{2} b^\mu \eta_{\mu\nu} b^\nu \right). \quad (30)$$

Therefore the $N = 2$ free Lagrangian is invariant under the 6 nilpotent fermionic generators, provided that $\lambda = \frac{1}{2}$.

The action admits the following quite remarkable set of equalities

$$\begin{aligned} \int dt \mathcal{L} &= \int dt \left(b^\mu \eta_{\mu\nu} b^\nu - b^\mu \eta_{\mu\nu} \dot{X}^\nu + \bar{\Psi}^\mu \dot{\Psi}^\nu \right) \\ &= \int dt Q \left(\bar{\Psi}^\mu \eta_{\mu\nu} (b^\nu - \dot{X}^\nu) \right) = - \int dt \bar{Q} \left(\Psi^\mu \eta_{\mu\nu} (b^\nu - \dot{X}^\nu) \right) \\ &= \int dt Q Q_V \left(\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu \right) = - \int dt \bar{Q} \bar{Q}_V \left(\Psi^\mu \eta_{\mu\nu} \bar{\Psi}^\nu \right) \\ &= \int dt Q_C \bar{Q} \left(\bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu \right) = - \int dt \bar{Q}_C Q \left(\Psi^\mu \eta_{\mu\nu} \dot{\bar{\Psi}}^\nu \right) \\ &= \int dt Q_C Q_V \left(\frac{1}{2} b^\mu \eta_{\mu\nu} b^\nu \right) = - \int dt \bar{Q}_C \bar{Q}_V \left(\frac{1}{2} b^\mu \eta_{\mu\nu} b^\nu \right). \end{aligned} \quad (31)$$

One can also check the Q_C invariance as follows.

$$Q_C Q_V Q \left(\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu \right) = \{Q_C, Q_V\} Q \left(\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu \right) + Q_V \left(Q Q_C \left(\bar{\Psi}^\mu \eta_{\mu\nu} \dot{\Psi}^\nu \right) \right). \quad (32)$$

For $\lambda = \frac{1}{2}$ the action of the bosonic symmetry $\{Q_C, Q_V\}$ on $Q(\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu)$ gives zero, modulo a time-derivative. Furthermore, for what concerns the second term in the r.h.s. of (32), we have

$$Q Q_C \left(\bar{\Psi}^\mu \eta_{\mu\nu} \Psi^\nu \right) = Q \left((t b^\mu - \frac{1}{2} X^\mu) \eta_{\mu\nu} \Psi^\nu \right) = Q \left(-\frac{1}{2} \Psi^\mu \eta_{\mu\nu} \Psi^\nu \right) = 0. \quad (33)$$

3.2 Interactions.

3.2.1 The prepotential.

The supersymmetric interaction is introduced in term of the “prepotential” $W[X^\mu]$.

A manifest Q -invariant term can be added to the action by setting

$$\mathcal{L}_{int} = Q \left(\bar{\Psi}^\mu \frac{\delta W}{\delta X^\mu} \right) = b^\mu \frac{\delta W}{\delta X^\mu} - \bar{\Psi}^\mu \frac{\delta^2 W}{\delta X^\mu \delta X^\nu} \Psi^\nu. \quad (34)$$

The full Q -invariant action is thus

$$\begin{aligned} \mathcal{S} &= \int dt \left(b^\mu \eta_{\mu\nu} b^\nu - b^\mu \eta_{\mu\nu} (\dot{X}^\nu + \frac{\delta W}{\delta X^\mu}) + \bar{\Psi}^\mu \eta_{\mu\nu} (\dot{\Psi}^\nu + \frac{\delta^2 W}{\delta X^\mu \delta X^\nu} \Psi^\nu) \right) \\ &\sim \int dt \left(-\frac{1}{4} \dot{X}^\mu \dot{X}^\nu - \frac{1}{4} \frac{\delta W}{\delta X^\mu} \eta_{\mu\nu} \frac{\delta W}{\delta X^\nu} + \bar{\Psi}^\mu \eta_{\mu\nu} (\dot{\Psi}^\nu + \frac{\delta^2 W}{\delta X^\mu \delta X^\nu} \Psi^\nu) \right). \end{aligned} \quad (35)$$

The Q , Q_V , \bar{Q} and \bar{Q}_V invariances, modulo a time derivative, of \mathcal{L}_{int} are warranted because

$$\mathcal{L}_{int} = Q \bar{Q} (W) = Q Q_V (W) = -\bar{Q} \bar{Q}_V (W). \quad (36)$$

3.2.2 The Q_C invariance.

The Q_C , \bar{Q}_C invariances, modulo a time derivative, of \mathcal{L}_{int} imply the following condition on the prepotential

$$\Psi^\mu \frac{\partial W}{\partial X^\nu} - X^\nu \frac{\partial^2 W}{\partial X^\mu \partial X^\nu} \Psi^\nu = 0 \Rightarrow \frac{\partial}{\partial X^\rho} \left(X^\mu \frac{\partial W}{\partial X^\mu} \right) = 0. \quad (37)$$

Therefore, the condition for having a Q_C -invariance is

$$X^\mu \frac{\partial W}{\partial X^\mu} = C, \quad (38)$$

whose general solution is

$$W = C \ln R + f \left(\frac{X^\mu}{R} \right). \quad (39)$$

C is an arbitrary constant and f is an arbitrary function of the non-dimensional quantities $\frac{X^\mu}{R}$, where $R^2 \equiv X^\mu \eta_{\mu\nu} X^\nu$.

This gives us a so-called conformal twisted supersymmetric quantum mechanics, often with possible topological observables. Its action can be untwisted to a Lagrangian that has ordinary conformal supersymmetry. For instance, for one particle on a plane with coordinates $X^i, i = 1, 2$, and for $f = 0$, one can select the conformal potential

$$\left(\frac{\partial W}{\partial X^i} \right)^2 = \frac{C^2}{R^2}. \quad (40)$$

It defines an interesting topological solvable quantum mechanics on the \mathbf{R}^{2*} plane with the origin excluded. The topological gauge function is then $\dot{X}^i = C \epsilon_{ij} \frac{X^j}{R^2}$, since

$$\left(\dot{X}^i + C \epsilon_{ij} \frac{X^j}{R^2} \right)^2 = \left(\dot{X}^i \right)^2 + \frac{C^2}{X^2} + 2C \dot{\theta}. \quad (41)$$

The corresponding $N = 2$ supersymmetric quantum mechanics mimics as an elementary model the topological Yang–Mills theory, which uses the selfduality equations as quantum field theory topological gauge functions [19].

Still keeping $i = 1, 2$, with $X \equiv X^1$ and $Y \equiv X^2$, the Q_C “topological” invariance also produces the superconformal quantum mechanics of a pair of particles evolving on a line with coordinates $X(t)$ and $Y(t)$ and interacting via a Calogero potential.

The latter is

$$\left(\frac{\partial W}{\partial X} \right)^2 + \left(\frac{\partial W}{\partial Y} \right)^2 = \frac{2}{(X - Y)^2} \quad (42)$$

and depends on the relative distance.

The prepotential W is indeed

$$W = \ln |X^1 - X^2| = \ln R + \ln \left| \frac{X^1 - X^2}{R} \right|. \quad (43)$$

It satisfies the general condition (38) for Q_C invariance with $C = 1$ everywhere apart from the unphysical (infinite energy) coincidence line $X^1 = X^2$ in \mathbf{R}^2 .

4 Curved target-space.

Let us come back to the action (2), built in [14]. The $\frac{1}{2}$ value of the coefficient in front of the Christoffel coefficient in the “topological gauge function” $g_{\mu\nu}(\dot{X}^\nu) - \frac{1}{2}\Gamma_{\mu,\rho\sigma}\bar{\Psi}^\rho\Psi^\sigma$ must be finely-tuned in order to get target-space covariance. For topological observables, defined from the cohomology of Q , this is not a critical issue, since they do not depend on the addition of Q -exact terms to the action.

The choice of this coefficient however, and thus the target-space covariance of the action, is implied by the additional requirement of the Q_V invariance. Indeed, one can easily check (by using the chain rule for the Q_V operator) that the above action can be expressed as a straightforward generalization of the flat space formula, this time with a coordinate-dependent metric $g_{\mu\nu}$. We get

$$\int dt L = QQ_V \int dt (\bar{\Psi}^\mu g_{\mu\nu} \Psi^\nu) = -\bar{Q}\bar{Q}_V \int dt (\bar{\Psi}^\mu g_{\mu\nu} \Psi^\nu). \quad (44)$$

Therefore, the requirement of invariance under both Q and Q_V implies the target-space covariance, which is an intriguing new result.

The Q_C invariance, on the other hand, is generally broken by terms which are proportional to derivatives of the metric. Thus, the compatibility between conformal invariance of the worldline and the target-space covariance is broken in the presence of a curved target-space metrics (see [7, 8] for a discussion of the conformal invariance with non-trivial backgrounds).

5 Higher-order Fermi field interactions.

A series of other of Q and Q_V invariant actions can be constructed by simply using the $B_{\mu_1\dots\mu_p}(X)$ forms of various degrees. They can be written as

$$\int dt L = QQ_V \int dt (\bar{\Psi}^\mu \Psi^\nu) (g_{\mu\nu}(X) + B_{\mu\nu}(X) + \sum_{p>1} B_{\mu\nu\mu_1\nu_1\dots\mu_p\nu_p}(X) \bar{\Psi}^{\mu_1} \Psi^{\nu_1} \dots \bar{\Psi}^{\mu_p} \bar{\Psi}^{\nu_p}). \quad (45)$$

Such actions are by construction Q and Q_V invariant. When one expands the above QQ_V -exact term, one finds a b dependence, such that higher Fermi field interactions do occur. Since the engineering dimensions of Ψ and $\bar{\Psi}$ adds to zero, there is no further limitation on the value of p , apart the relation $2p \leq d$, due to the fact that Ψ^μ and $\bar{\Psi}^\nu$ are anticommuting fields. The forms $B_{\mu\nu}$ are analogous to the Kalb–Ramond fields.

6 The twist transformation.

We discuss now the relation between the twisted $N = 2$ SCA \mathcal{G}^\sharp and the ordinary $N = 2$ SCA (the $sl(2|1)$ superalgebra). Their D -module representations are given in Appendix **A** and **B**,

respectively.

Both superalgebras possess a maximal common subalgebra \mathcal{B} , which is made of 5 (3 even and 2 odd) generators, defined as follows (in the right hand side of the equations we present the combinations in terms of the \mathcal{G}^\sharp generators),

$$\mathcal{B} = \{D = \frac{1}{2}(S - \bar{S}), \quad H, \quad W = Z + \bar{Z}, \quad Q_1 = Q + Q_V - \bar{Q}, \quad Q_2 = \bar{Q} - \bar{Q}_V - Q\}. \quad (46)$$

The superalgebra \mathcal{B} , $\mathcal{G}^\sharp \supset \mathcal{B} \subset sl(2|1)$, is the Borel subalgebra of $sl(2|1)$ given by the Cartan and the negative-root generators.

One should note that no “conformal generator” (i.e., carrying an explicit dependence on t) belongs to both \mathcal{G}^\sharp and $sl(2|1)$. This has a consequence. In order to introduce a superconformal symmetry we are faced with two mutually exclusive paths. Either we impose the invariance under the ordinary $N = 2$ superconformal algebra (ending up with an $sl(2|1)$ -invariance), or we impose the invariance under the twisted generators (ending up with the $N = 2$ twisted superconformal algebra \mathcal{G}^\sharp).

Let us denote respectively as \mathcal{L} and \mathcal{L}^\sharp both invariant free Lagrangian for the ordinary $N = 2$ SCA and the twisted $N = 2$ SCA. It is convenient to introduce different notations for the component fields entering the ordinary and the twisted $(1, 2, 1)$ supermultiplets. For the ordinary $N = 2$ SCA let us have the $(Y(t); \xi_1(t), \xi_2(t); g(t))$ component fields and for the twisted $N = 2$ SCA the $(X(t); \Psi(t), \bar{\Psi}(t); b(t))$ component fields.

With a convenient normalization we can present the free Lagrangians as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \left(\dot{Y}^2 + g^2 - \xi_1 \dot{\xi}_1 - \xi_2 \dot{\xi}_2 \right), \\ \mathcal{L}^\sharp &= b^2 - b \dot{X} + \bar{\Psi} \dot{\Psi}. \end{aligned} \quad (47)$$

In order to identify them ($\mathcal{L}^\sharp = \mathcal{L}$), we have to provide the invertible “twist transformation” T , which link both sets of fields. We have

$$\begin{aligned} X &= \sqrt{2}iY, \\ b &= \frac{1}{\sqrt{2}}(g + i\dot{Y}), \\ \Psi &= \frac{1}{\sqrt{2}}(\xi_1 - \xi_2), \\ \bar{\Psi} &= \frac{i}{\sqrt{2}}(\xi_1 + \xi_2), \end{aligned}$$

or, in matrix form,

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 2i & 0 & 0 & 0 \\ i\partial_t & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & i & i \end{pmatrix}, \quad T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & 0 & 0 \\ -\partial_t & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -1 & -i \end{pmatrix}. \quad (48)$$

The above twist transformation T is intrinsically complex. No solution for T can be found within the real numbers. Therefore T only makes sense if the component fields $X, Y, b, g, \Psi, \bar{\Psi}, \xi_1, \xi_2$ are taken as complex fields. We can introduce a reality condition on the twisted fields $X, b, \Psi, \bar{\Psi}$ or a reality condition on the ordinary fields Y, g, ξ_1, ξ_2 . Both reality conditions, however, are mutually incompatible.

This feature of the twist transformation does not make it less useful. Indeed, to recover, via path integral, the ordinary correlation functions from the twisted correlation functions (or vice-versa), the only needed tool is an analytical extension. This allows to perform the twist transformation (or its inverse).

In the Conclusions we comment more on the implications of the relation between ordinary and twisted formulations for extended supersymmetric theories.

Let us mention here that the superalgebra \mathcal{G}^\sharp could be further enlarged by adding the even conformal generator K (which, together with H and D , closes the $sl(2)$ conformal algebra) and all the extra generators which are required to close the (anti)commutation relations. Such an enlarged superalgebra is of little significance since it does not produce any further information and constraint on the superconformal invariance, besides those obtained from \mathcal{G}^\sharp and its relevant subalgebras.

7 Conclusions and outlook.

In this work we proved that the $N = 2$ superconformal quantum mechanics based on the $(1, 2, 1)$ balanced supermultiplets admits a twisted formulation, controlled by a twisted superalgebra. The twisted superalgebra \mathcal{G}^\sharp contains 6 nilpotent odd generators and 6 even generators (including a central charge). The fermionic generators can be used to define BRST-type cohomologies. The invariance under different subalgebras determine different types of models. For some of them, the invariance under just a pair of the 6 generators implies the invariance under some of the other ones. For instance, the invariance under just the pair of nilpotent generators Q and Q_V implies a \bar{Q}, \bar{Q}_V -invariant supersymmetric quantum mechanics for any given potential in a flat target space; for a curved target-space, the request of both Q and Q_V invariance determines, after solving the equations of motion of the auxiliary fields, a supersymmetric action which is covariant, being expressed in terms of the metric, the vectors $X, \Psi, \bar{\Psi}$, the covariant derivatives and the Riemann curvature.

A striking observation concerns the fact that the invariance under Q and \bar{Q}_C , whose anticommutator closes on a central charge, together with the property that one is considering Lagrangians with ghost number zero and standard dimension, completely determines the superconformally actions with all their extra symmetries.

These features share many features and might be at the root of recent results in higher dimensions, for both twisted super-Yang–Mills theories [18] and supergravity [20]. These quan-

tum field cases are much more constrained, due to the presence of additional symmetries such as the Lorentz symmetry, the R-symmetry, etc. The key point is that the twisted formulation reduces in a controllable way the size of the symmetries, keeping however the symmetry algebra large enough to uniquely determine the theory⁷. Therefore, and quite remarkably, once the theory has been defined by this smaller set of generators, one discovers that it possesses more symmetries, which appear “for free”. The presence of these extra symmetries eventually allows to untwist the fermions, so that one is able to recover the physical spin-statistic relation.

For the twisted $N = 4$ maximal superYang-Mills theory [21, 22, 15] one can for instance covariantly select, among its 16 supersymmetry generators, $n \leq 9$ generators [23] that close “off-shell” (twisted subalgebras can be directly constructed and regarded as a germ for the superPoincaré invariance, see [24, 25]). Moreover, only 6 among the 9 generators are needed to define the theory, either in the $N = 4$, $d = 4$ as well as in the $N = 1$, $d = 10$ SYM cases. Furthermore, one can enlarge the twist from the super-Poincaré algebra to the super-conformal case [26]. Within this framework one finds that the invariance under 4 twisted fermionic generators mixing sectors of both Poincaré and conformal supersymmetry fully determines the theory [18].

In the setting of the $N = 2$ supersymmetric mechanics we proved that the two fermionic generators Q and \bar{Q}_C , together with a central charge, contain enough information to determine the $N = 2$ superconformal action with its full set of fermionic and bosonic symmetries. We may also mention the recently found example of the $N = 1$, $d = 4$ supergravity, which is very well-known in the untwisted formalism, and can be therefore used as a safe playground for exploring non-trivial properties of the twist procedure. It can be obtained by using only a $U(2) \subset SO(4)$ symmetry, in analogy with the $N = 1$, $d = 4$ super-Yang-Mills theory. Only $3 = 1 + 2$ global supersymmetries are needed to define, modulo boundary terms, the supergravity. The fourth susy generator is implied by the 3 former ones and the full $SO(4)$ symmetry is finally found after the untwisting.

As a consequence of the seemingly general existence of the twist, burdening quantum field theory questions, such as the existence of supercharges which only close modulo the use of some equations of motion may well just become irrelevant since, in the twisted formulation, one discovers that the theory is determined and controlled by a subset of supercharges which close off-shell. The Poincaré (as well as the conformal) supersymmetry might thus appear as a kind of physical (and over-determining) effective symmetry, which sometimes emerges after untwisting a TQFT, whenever it is possible.

As a closing remark we wish to point out that we made explicit the connection between

⁷The twisted theories possess normal local Feynmann propagators. Therefore, their simplification cannot be compared with that obtained by choosing a light-cone gauge. In the latter case some simplicity is gained, but the local properties of the theory are harmed, due to a propagator with cut singularities which cannot be handled in a satisfactory way. Twisted theories are better behaved.

twisted and ordinary $N = 2$ superconformal quantum mechanics, determining the twisted superalgebra (and its D -module representation) associated with the twisted invariance. A natural future step consists in investigating the relation between ordinary and twisted $N = 4$ superconformal quantum mechanics. In its “ordinary” side [27, 28, 29, 30], the $N = 4$ superconformal algebra (which plays the role of the $sl(2|1)$ $N = 2$ SCA) is the exceptional superalgebra $D(2, 1; \alpha)$. An open question is to determine its twisted superalgebra counterpart.

A natural motivation to investigate twisted superconformal quantum mechanics with large N comes from the geometric Langlands program which can be described [31] as a twisted $N = 4$ Super-Yang–Mills theory compactified in $2D$. It is expected that a world-line method based on the one-dimensional twisted superconformal mechanics can be employed to reconstruct the two-dimensional theory.

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Appendix A: the D -module representation of the twisted $N = 2$ superconformal algebra.

For completeness we present here the D -module representation of the twisted $N = 2$ superconformal algebra \mathcal{G}^\sharp . It is a \mathbf{Z}_2 -graded Lie algebra $\mathcal{G}^\sharp = \mathcal{G}_0^\sharp \oplus \mathcal{G}_1^\sharp$, with the even sector \mathcal{G}_0^\sharp given by the generators $H, c, S, \bar{S}, Z, \bar{Z}$ and the odd sector \mathcal{G}_1^\sharp given by the generators $Q, \bar{Q}, Q_V, \bar{Q}_V, Q_C, \bar{Q}_C$.

The D -module representation consists of 4×4 supermatrices acting on the supermultiplet $\begin{pmatrix} X(t) \\ b(t) \\ \Psi(t) \\ \bar{\Psi}(t) \end{pmatrix}$, whose two upper component fields are bosonic and the two lower component fields are fermionic. It is convenient to leave arbitrary the real parameter λ . The closure of the (anti)commutation relations is not affected by it. As discussed in the main text, see formula (17), the applicability of the \mathcal{G}^\sharp D -module representation to the superconformal invariance forces us to set $\lambda = \frac{1}{2}$.

In the expressions below we set for convenience $\partial_t := \frac{d}{dt}$. We have

$$\begin{aligned}
Q &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \bar{Q} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
Q_V &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \partial_t \\ \partial_t & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \bar{Q}_V &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 0 \\ -\partial_t & 1 & 0 & 0 \end{pmatrix}, \\
Q_C &= \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & t & 0 & 0 \end{pmatrix}, & \bar{Q}_C &= \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & \lambda \\ \lambda & -t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
H &= \begin{pmatrix} \partial_t & 0 & 0 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & \partial_t \end{pmatrix}, & c &= \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},
\end{aligned} \tag{49}$$

$$S = \begin{pmatrix} t\partial_t - \lambda & 0 & 0 & 0 \\ 0 & t\partial_t + 1 - \lambda & 0 & 0 \\ 0 & 0 & t\partial_t + 1 - \lambda & 0 \\ 0 & 0 & 0 & t\partial_t - \lambda \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} -t\partial_t + \lambda & 0 & 0 & 0 \\ 0 & -t\partial_t + \lambda - 1 & 0 & 0 \\ 0 & 0 & -t\partial_t + \lambda & 0 \\ 0 & 0 & 0 & -t\partial_t + \lambda - 1 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{Z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The (anti)-commutation relations are presented in Section 2 (formulas (6),(12) and (13)).

Appendix B: the D -module representation of $sl(2|1)$.

In order to allow the comparison with the twisted case we present here the D -module representation of the $N = 2$ superconformal algebra $sl(2|1)$ acting on a $(1, 2, 1)$ supermultiplet whose component fields have the same engineering dimensions as in the twisted case. As before, a free parameter λ is allowed. The application of the transformations to superconformal invariant actions forces us to set $\lambda = \frac{1}{2}$.

The D -module representation can be obtained by closing the superalgebra recovered by applying the $sl(2)$ D -module representation to the D -module representation, given in [32, 33], of the global $N = 2$ supercharges.

We end up with the following set of even (H, W, D, K) and odd $(Q_1, Q_2, \tilde{Q}_1, \tilde{Q}_2)$ generators closing the (anti)commutation relations

$$[D, H] = -H, \quad [D, K] = K, \quad [K, H] = 2D, \quad (50)$$

together with

$$\begin{aligned} \{Q_1, Q_1\} &= \{Q_2, Q_2\} = 2H, \\ \{\tilde{Q}_1, \tilde{Q}_1\} &= \{\tilde{Q}_2, \tilde{Q}_2\} = -2K, \\ \{Q_1, \tilde{Q}_1\} &= \{Q_2, \tilde{Q}_2\} = 2D, \\ \{Q_1, \tilde{Q}_2\} &= \{\tilde{Q}_1, Q_2\} = W \end{aligned} \quad (51)$$

and

$$\begin{aligned} [H, \tilde{Q}_i] &= Q_i, & [K, Q_i] &= \tilde{Q}_i, \\ [D, Q_i] &= -\frac{1}{2}Q_i, & [D, \tilde{Q}_i] &= \frac{1}{2}\tilde{Q}_i, \\ [W, Q_i] &= -\epsilon_{ij}Q_j, & [W, \tilde{Q}_i] &= -\epsilon_{ij}\tilde{Q}_j, \end{aligned} \quad (52)$$

for $i = 1, 2$ ($\epsilon_{12} = -\epsilon_{21} = 1$).

Their explicit expression is given by

$$H = \begin{pmatrix} \partial_t & 0 & 0 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & \partial_t \end{pmatrix},$$

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} t\partial_t - \lambda & 0 & 0 & 0 \\ 0 & t\partial_t + 1 - \lambda & 0 & 0 \\ 0 & 0 & t\partial_t + \frac{1}{2} - \lambda & 0 \\ 0 & 0 & 0 & t\partial_t + \frac{1}{2} - \lambda \end{pmatrix},$$

$$K = \begin{pmatrix} -t^2\partial_t + 2\lambda t & 0 & 0 & 0 \\ 0 & -t^2\partial_t + (2\lambda - 2)t & 0 & 0 \\ 0 & 0 & -t^2\partial_t + (2\lambda - 1)t & 0 \\ 0 & 0 & 0 & -t^2\partial_t + (2\lambda - 1)t \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \partial_t \\ \partial_t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\partial_t & 0 \\ 0 & -1 & 0 & 0 \\ \partial_t & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{Q}_1 = \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & 0 & t\partial_t - 2\lambda + 1 \\ t\partial_t - 2\lambda & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix},$$

$$\tilde{Q}_2 = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & -t\partial_t + 2\lambda - 1 & 0 \\ 0 & -t & 0 & 0 \\ t\partial_t - 2\lambda & 0 & 0 & 0 \end{pmatrix}. \quad (53)$$

References

- [1] S. Fubini, E. Rabinovici, *Superconformal Quantum Mechanics*, Nucl. Phys. **B245** (1984) 17.
- [2] E. Ivanov, S. Krivonos, V. Leviant, *Geometric superfield approach to superconformal mechanics*, J. Phys. A: Math. Gen **22** (1989) 4201.
- [3] D. Z. Freedman, P. Mende, *An Exactly Solvable N Particle System in Supersymmetric Quantum Mechanics*, Nucl. Phys. **B344** (1990) 317.
- [4] P. Claus, M. Derix, R. Kallosh, J. Kumar, P. Townsend, A. van Proeyen, *Black Holes and Superconformal Mechanics*, Phys. Rev. Lett. **81** (1998) 4553; hep-th/9804177.
- [5] J. A. de Azcarraga, J. M. Izquierdo, J. C. Perez Bueno, P. K. Townsend, *Superconformal mechanics, black holes, and non-linear realizations*, Phys. Rev. **D 59** (1999) 084015; hep-th/9810230.
- [6] N. Wyllard, *(Super)conformal many-body quantum mechanics with extended supersymmetry*, J. Math. Phys. **41** (2000) 2826; hep-th/9910160.
- [7] R. Britto-Pacumio, J. Michelson, A. Strominger, A. Volovich, *Lectures on superconformal quantum mechanics and multi-black hole moduli spaces*, in Progress in String Theory and M-Theory, NATO Science Series Vol. 564 (2001) 235; hep-th/9911066.
- [8] G. Papadopoulos, *Conformal and Superconformal Mechanics*, Class. Quant. Grav. **17** (2000) 3715; hep-th/0002007.
- [9] A. Galajinsky and O. Lechtenfeld, *Harmonic N = 2 Mechanics*, Phys. Rev. **D 80** (2009) 065012; arXiv:0907.2242.
- [10] E. Witten, *Supersymmetry and Morse Theory*, J. Diff. Geom. **17** (1982) 661.
- [11] V. P. Akulov and A. I. Pashnev, *Quantum Superconformal Model In (1,2) Space*, Theor. Math. Phys. **56** (1983) 862 (Teor. Mat. Fiz. **56** (1983) 344).
- [12] L. Baulieu, B. Grossman, *A topological interpretation of stochastic quantization*, Phys. Lett. **B212** (1988) 351.

- [13] D. Birmingham, M. Rakowski, G. Thompson, *Topological field theories, Nicolai maps and BRST quantization*, Phys. Lett. **B214** (1988), 381.
- [14] L. Baulieu, I. M. Singer, *The Topological Sigma Model*, Comm. Math. Phys. **125** (1989) 227.
- [15] L. Baulieu, F. Toppan, *One-Dimensional Structures Behind Twisted and Untwisted Super-Yang-Mills Theory*, Lett. Math. Phys. DOI: 10.1007/s11005-011-0505-x (Online First); (2011), arXiv:1010.3004.
- [16] L. Frappat, A. Sciarrino, P. Sorba, *Dictionary On Lie Algebras And Superalgebras*, Acad. Press (2000); hep-th/9607161.
- [17] V. de Alfaro, S. Fubini, G. Furlan, *Conformal invariance in quantum mechanics*, Nuovo Cim. **A34** (1976) 569.
- [18] L. Baulieu, *Four twisted differential operators for the $N=4$ superconformal algebra*, Nucl. Phys. B845, 436, 2011, (hep-th/1009.6176).
- [19] L. Baulieu, E. Rabinovici, *On the calculability of observables in topological quantum mechanical models*, Phys. Lett. **B316** (1993) 93.
- [20] L. Baulieu, M. Bellon, V. Reys, *$N=1$ twisted supergravity*. To appear.
- [21] C. Vafa and E. Witten, *A strong coupling test of S -duality*, Nucl. Phys. **B431** (1994) 3; hep-th//9408074.
- [22] N. Marcus, *The other topological twisting of $\mathcal{N} = 4$ Yang-Mills*, Nucl. Phys. **B 452** (1995) 331; hep-th/9506002.
- [23] F. Toppan, *Irreps and Off-shell Invariant Actions of the N -Extended Supersymmetric Quantum Mechanics*, POS IC (2006) 033; hep-th/0610180.
- [24] L. Baulieu, G. Bossard, *New Results on $N=4$ SuperYang-Mills Theory*, Phys. Lett. **B632** (2005) 131; hep-th/0507004.
- [25] L. Baulieu, G. Bossard, *Finiteness Properties of the $N=4$ Super-Yang-Mills Theory in Supersymmetric Gauge*, Nucl. Phys. **B753** (2006) 252; hep-th/0605164.
- [26] P. Medeiros, J. Figueroa-O'Farrill, C. Hull, B. Spence, *Conformal topological Yang-Mills theory and de Sitter holograph*, JHEP 0208 (2002) 055; arXiv:hep-th//0111190.
- [27] E. E. Donets, A. Pashnev, J. Juan Rosales, M. M. Tsulaia, *$N = 4$ Supersymmetric Multidimensional Quantum Mechanics, Partial SUSY Breaking and Superconformal Quantum Mechanics*, Phys. Rev. **D61** (2000) 043512; hep-th/9907224.

- [28] E. Ivanov, S. Krivonos, O. Lechtenfeld, *New variant of $N = 4$ superconformal mechanics*, JHEP 0303 (2003) 014; hep-th/0212303.
- [29] E. Ivanov, S. Krivonos, O. Lechtenfeld, *$N = 4$, $d = 1$ supermultiplets from nonlinear realizations of $D(2, 1; \alpha)$* , Class. Quantum Grav. 21 (2004) 1031; hep-th/0310299.
- [30] S. Krivonos, O. Lechtenfeld, *Many-particle mechanics with $D(2, 1; \alpha)$ superconformal symmetry*, JHEP 1102 (2011) 042; arXiv:1012.4639.
- [31] A. Kapustin, E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program*, Comm. Numb. Theory Phys. 1 (2007), 1; hep-th/0604151.
- [32] A. Pashnev and F. Toppan, *On the Classification of N -Extended Supersymmetric Quantum Mechanical Systems*, J. Math. Phys. 42 (2001), 5257; (hep-th/0010135).
- [33] Z. Kuznetsova, M. Rojas, F. Toppan, *Classification of irreps and invariants of the N -extended Supersymmetric Quantum Mechanics*, JHEP 0603 (2006), 098; (hep-th/0511274).